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Stability of Potential Systems to General Positional Perturbations

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This paper shows that a potential system with one multiple eigenvalue of multiplicity two or more can assuredly be made unstable by infinitesimal positional perturbations. The explicit nature of these pertubatory forces is provided. It is shown that the matrices that describe these perturbatory forces are not required to commute with the potential matrix. The general positional perturbatory forces that bring about instability in the perturbed potential system are shown to include, as special cases, circulatory forces that do and that do not commute with the potential matrix. The condition on commutativity of matrices is quite stringent, and its removal has a significant effect on generalizing the conditions that lead to instability. The paper expands the generalized Merkin instability result to include more general, noncirculatory positional perturbations, and it eliminates restrictions imposed on the perturbations by commutation requirements. The structure of infinitesimal as well as finite perturbatory matrices that guarantee instability is obtained. Practical implications of the mathematical results to natural and engineered systems are given. It is pointed out that potential systems with two "nearly" equal frequencies of vibration are, in general, susceptible to instability, created by "small" perturbatory forces, where the terms nearly and small are quantified.

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Nomenclature

A_p	=	p-by-p real matrix
a, b, c	=	real numbers
B	=	three-by-three real matrix
B_r	=	r-by-r real matrix
C	=	three-by-two real matrix
C_r	=	r-by-p real matrix
k	=	stiffness (positive number)
$\tilde{M}, \tilde{K}, K$	=	n-by-n real matrices
m	=	multiplicity of eigenvalue λ
$\hat{N}, N$	=	n-by-n real skew-symmetric perturbation matrices
$\bar{N}_p$	=	p-by-p real skew-symmetric matrix
$\tilde{P}, \hat{P}, P$	=	n-by-n real perturbation matrix
q, x, y	=	real n vectors
r	=	$n - p$
T	=	n-by-n real orthogonal matrix
t_λ^i, t^j	=	orthonormal column vectors of matrix T
z_p	=	real p vector
z_r	=	real r vector
α	=	real nonzero number
$\delta, \beta, \varepsilon, \gamma, \eta$	=	real numbers
θ	=	α/k
$\hat{\Lambda}, \Lambda_r$	=	diagonal matrices
λ	=	multiple eigenvalue of K
μ	=	λ^2
ρ	=	$\omega_1 - \omega_2$
ω_1, ω_2	=	frequencies of vibration

I. Introduction

POTENTIAL systems are important in physics and engineering because most real-life systems (those that occur naturally and those that are engineered) are usually modeled as potential systems. After linearization, the stability of such systems to perturbations has therefore been a question of considerable and long-standing interest to the physics, engineering, and mathematics communities. Merkin discovered a very important and remarkable result that states that, for a potential system (all of whose frequencies of vibration are identical), the addition of any circulatory force causes the system to become unstable [1]. Circulatory force additions to potential systems arise in many real-life applications ranging from aerospace structures and aeroelasticity to brake squeal, wheel shimmy, and bipedal motion [2–7]. Potential forces are represented in linear systems by a symmetric stiffness matrix (which we shall often call the potential matrix, for short), and systems that have only potential forces are called potential systems. Circulatory forces are represented by skew-symmetric matrices, which we shall often call circulatory matrices.

This important result obtained by Merkin [1] was extended by Bulatovic to say that, if the potential matrix and the circulatory (skew-symmetric) matrix commute, then the addition of a circulatory matrix to the potential system causes it to become unstable [8]. Reference [8] however, does not provide physical insights or address questions of practical importance in physics and engineering, namely, whether multiplicity of a few (or even a single one) of the frequencies of vibration of a potential system would cause the addition to it of a commuting circulatory matrix to render the system unstable. Recently, such a generalization of Merkin's result [1] was provided, giving insights into the role played by the extent and number of multiple frequencies of the potential system [9]. It shows that, when the circulatory and the potential matrices commute, the potential matrix must have at least two frequencies that are identical; and that when two (or more) frequencies of the potential system are identical, then the system becomes unstable when an appropriate infinitesimal circulatory matrix, which commutes with the potential matrix, is added to the potential system. It is this central idea that at least two frequencies of the potential system must be identical when the circulatory matrix and the potential matrix commute [9] that is central to this paper and expanded upon here.

The requirement for commutation of the circulatory matrix with the potential matrix is a fairly strong condition. When the potential matrix and the circulatory matrix commute, we use the insights provided by the results in Ref. [9], which point out that it is these multiple eigenvalues of the potential system that are at the root of

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being able to find (infinitesimal) perturbatory circulatory matrices that make such potential circulatory systems unstable. In this paper, we show that this condition of commutativity is not necessary as long as the potential matrix has at least one multiple eigenvalue. Starting with the assumption of multiple eigenvalues of the potential system, the general structure of positional perturbatory matrices that make the potential system unstable and that do not need to commute with the potential (symmetric) matrix of the system is obtained. Routes that make the perturbed potential system unstable in the presence of multiple eigenvalues of the potential system are explored. The compass of matrices whose addition causes such potential systems to become unstable is expanded beyond those that are only circulatory and those that commute with the potential matrix. It is shown that the set of these general (infinitesimal) perturbatory matrices, which includes circulatory matrices that do and do not commute with the potential matrix, when added to the potential system, guarantee that their addition leads to instability.

Reference [2] gives a different approach from that given in this paper. It considers singularities of the stability boundary of systems of the form $\ddot{x} + Px = 0$, where the matrix P is nonsymmetric and depends on parameters. Points in the parameter space at which the system has multiple imaginary eigenvalues correspond to singularities of the boundary. It is shown that, when P depends on three parameters and has a semisimple imaginary eigenvalue with multiplicity two, the region of instability lies inside a cone with its apex at the singular point. The equation of this cone then provides a quantitative measure of the unstable region. In essence, this method is based on the first approximation of perturbed eigenvalues, and therefore considers small perturbations. The approach presented in this paper is simpler and more direct. It is applicable to systems with imaginary eigenvalues that can have an arbitrary number of multiplicities, and it includes positional forces that can have finite intensities.

Consider the potential system described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q = 0 \tag{1}$$

where the n-by-n matrix $\tilde{M}$ is a positive definite matrix, and $\tilde{K}$ is a real symmetric matrix. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation. The addition of a perturbing positional force to this system results in the system described by the equation

$$\tilde{M}\ddot{q} + (\tilde{K} + \tilde{P})q = 0 \tag{2}$$

where $\tilde{P}$ is a real matrix. Making the transformation $q(t) = \tilde{M}^{-1/2}x(t)$ and premultiplying Eqs. (1) and (2) by $\tilde{M}^{-1/2}$, we get the following equations that describe the potential system and the perturbed potential system:

$$\ddot{x} + Kx = 0 \tag{3}$$

$$\ddot{x} + (K + P)x = 0 \tag{4}$$

where the symmetric matrix $K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$, and $P = \tilde{M}^{-1/2}\tilde{P}\tilde{M}^{-1/2}$. Clearly, system (2) is equivalent to system (4), and we shall from here on consider this system.

The generalized Merkin result states that, when P is skew-symmetric, and K and P commute, K necessarily has multiple eigenvalues, and the potential system given by Eq. (1) is unstable in the presence of such a perturbatory skew-symmetric matrix P [9].

This paper deals with the situation where the potential system described by the matrix K has multiple eigenvalues, and it intends to expand the set of matrices that makes the perturbed potential system [Eq. (4)] unstable beyond those that are only circulatory and commute with the matrix K. It also aims to obtain a further expansion to more general positional perturbatory matrices that are not necessarily skew-symmetric, that are not required to commute with K, and that guarantee instability of the perturbed potential system. Although the results obtained are valid for symmetric matrices, from practical considerations, we are interested in 1) stable potential systems in which $K > 0$, and 2) their susceptibility to being destabilized through the addition of positional perturbatory matrices P [Eq. (4)].

To motivate the results obtained in this paper, let us consider the simple five-degree-of-freedom potential system described by

$$\ddot{x} + Kx = 0 \tag{5}$$

where $x = [x_1, x_2, x_3, x_4, x_5]^T$. The potential matrix K is given by

$$K = \operatorname{diag}(k_1, k_1, k_3, k_4, k_5) \tag{6}$$

with, $k_i > 0$, $i = 1, 3, 4, 5$. The eigenvalue k_1 of the matrix K has multiplicity two. Furthermore, we assume that $k_i \neq k_j$ for $i \neq j$; we shall relax this condition later on.

Consider the real perturbatory matrix

$$p = \left[\begin{array}{cc:ccc} 0 & \alpha & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 \\ \hdashline c_{11} & c_{12} & b_{11} & b_{12} & b_{13} \\ c_{21} & c_{22} & b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & b_{31} & b_{32} & b_{33} \end{array}\right] := \begin{bmatrix} \bar{N}_2 & 0 \\ C & B \end{bmatrix} \tag{7}$$

where $\alpha \neq 0$ is any arbitrary (real) number. In the second preceding equality, we denote the two-by-two skew-symmetric matrix in the upper left corner of P by $\bar{N}_2$, as well as the block matrices B and C as shown, which have (real) arbitrary elements in them.

The potential system described by Eq. (5) when perturbed by a positional force described by the matrix P in Eq. (7) is described by

$$\ddot{x} + Kx + Px = 0, \quad \text{or} \quad \ddot{x} + \underbrace{\begin{bmatrix} k_1 I_2 & 0 \\ 0 & \tilde{k} \end{bmatrix}}_{K} x + \underbrace{\begin{bmatrix} \bar{N}_2 & 0 \\ C & B \end{bmatrix}}_{P} x = 0 \tag{8}$$

where $\tilde{k} = \operatorname{diag}(k_3, k_4, k_5)$. Note that the matrices K and P do not, in general, commute; although, the matrices $k_1 I_2$ and $\bar{N}_2$ do.

On a closer look at Eq. (8), we observe the following:

1) The commutation of the matrices $k_1 I_2$ and $\bar{N}_2$ in the upper block is caused by the fact k_1 is a multiple eigenvalue of the matrix K, and the first two equations of motion pertain to this multiple eigenvalue of K of the potential system.

2) The uncoupled two-degree-of-freedom potential subsystem involves only these first two equations of motion [see Eq. (5)], as well as only the coordinates $x_1(t)$ and $x_2(t)$. As seen in Eq. (8), it is subjected to a nonzero (perturbatory) circulatory force provided by the skew-symmetric matrix $\bar{N}_2$. Furthermore, this perturbatory force leaves this subsystem still uncoupled from the remainder of the five-degree-of-freedom system. This subsystem formed by the first two equations (the upper block) in the set of equations given in Eq. (8) is

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{z}_2} + k_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{k_1 I_2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{z_2} + \underbrace{\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}}_{\bar{N}_2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{z_2} = 0 \tag{9}$$

The upper two-by-two block $k_1 I_2$ and the skew-symmetric matrix $\bar{N}_2$ that multiplies the 2-vector $z_2(t) := [x_1 \quad x_2]^T$ commute [see Eq. (9)], and therefore this two-degree-of-freedom subsystem that describes the time evolution of the coordinate $z_2(t)$ is unstable [8,9]. The instability is a flutter instability. Notice that the reason that $k_1 I_2$ commutes with $\bar{N}_2$ is that the eigenvalues (frequencies of vibration) of the matrix K (potential system) are identical, as pointed out in Eq. (1). And, it is this two-degree-of-freedom unstable subsystem, which is a component of the entire five-degree-of-freedom system described in Eq. (8), that then renders the entire system unstable.

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3) Thus, irrespective of the dynamics of the "remainder" of the system (namely, the dynamics of $z_r(t) \coloneqq [x_3 \quad x_4 \quad x_5]^T$, with subscript "$r$" for remainder), we are assured of the instability of Eq. (8) because its unstable behavior resides in the subcomponent $z_2(t)$ of the column vector $x(t) = [z_2(t) \quad z_r(t)]^T$, and the evolution in time of this unstable subcomponent $z_2(t)$ is uncoupled from that of the remainder of the system. Thus, no matter what matrices B and C may be, system (8) is assuredly unstable.

4) In the special case when $C = 0$, the matrix P in Eq. (7) reduces to

$$P \coloneqq \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} \bar{N}_2 & 0 \\ 0 & B \end{bmatrix} \tag{10}$$

which is a direct sum and can be written as

$$P = \bar{N}_2 \oplus B \tag{11}$$

In Eq. (11), the matrix $\bar{N}_2$ is any nonzero two-by-two skew-symmetric matrix, and the lower block-diagonal three-by-three matrix B is an arbitrary three-by-three matrix. The matrices P and K do not commute, in general, since B and $\tilde{k}$ do not [see Eq. (8)]. As before, the subsystem described by the coordinate $z_2(t)$ continues to be unstable, exhibiting a flutter instability, and thereby makes the entire system described by Eq. (8) unstable.

5) When $C = 0$ and B is skew-symmetric, then the matrix P also becomes skew-symmetric and represents a circulatory force. Denoting this skew-symmetric matrix P by N, it can be written as

$$N \coloneqq \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta & -\delta \\ 0 & 0 & \beta & 0 & -\gamma \\ 0 & 0 & \delta & \gamma & 0 \end{bmatrix} = \begin{bmatrix} \bar{N}_2 & 0 \\ 0 & \bar{N}_3 \end{bmatrix} \tag{12}$$

in which the skew-symmetric matrix $\bar{N}_3 \coloneqq B$ has parameters β, δ, and γ that are arbitrary (real) numbers. When these parameters are nonzero, the skew-symmetric matrix N does not commute with K because the matrix $\bar{N}_3$ does not commute with the matrix $\tilde{k}$ [see Eq. (8)]. With this circulatory matrix $N = P$ in Eq. (8), the subsystem described by the coordinate $z_2(t)$ continues to be unstable, thereby making the entire system unstable.

Our example illustrates that infinitesimal circulatory matrices like N in Eq. (12) that perturb a potential system cause instability when the potential system has a single multiple eigenvalue without any requirement that these circulatory matrices commute with the potential matrix. Note that N is a special case to which the matrix P [Eq. (7)] reduces when the arbitrary matrix $C = 0$ and the arbitrary matrix B is skew-symmetric.

Furthermore, since B is arbitrary, the matrix N includes those perturbatory matrices that do commute with K and are circulatory. To see this, simply set $B = 0$ in Eq. (12); now, the matrices K and N commute.

6) More importantly, we observe that, no matter what the dynamics of the remainder of the system described by the coordinate $z_r(t)$ might be, the subsystem described by Eq. (9) is unstable. Hence, it does not matter what the other eigenvalues k_3, k_4, and k_5 of the potential matrix K are because these parameters only affect the dynamics of the remainder of the system; they have no effect on the unstable uncoupled subsystem described by the coordinate $z_2(t)$. Hence, these three parameters, which describe the matrix K, could have any values we wish, and some of them might even be allowed to take to the value of k_1 (something we did not allow initially but had promised to relax); in which case, of course, the eigenvalue k_1 of the potential system (matrix K) would no longer have a multiplicity of just two, but higher. So, no matter what the multiplicity of an eigenvalue of K might be, as long as it is greater than two, we can create a two-degree-of-freedom subsystem with just two of the (possibly more than two) identical eigenvalues, use an arbitrary nonzero two-by-two skew-symmetric matrix $\bar{N}_2$, and uncouple it from the rest of the system as in Eq. (8). That would make this two-degree-of-freedom subsystem unstable, which would then be sufficient to make the entire system unstable.

7) We chose the skew-symmetric matrix $\bar{N}_2$ in Eqs. (7) and (8) so that the subsystem shown in Eq. (9) is guaranteed to be unstable [8,9]. But, in fact, we could have chosen (instead of the skew-symmetric matrix $\bar{N}_2$) any two-by-two real matrix A_2 whose eigenvalues are complex with a nonzero imaginary part. This is because the replacement of $\bar{N}_2$ by such a matrix A_2 in Eq. (9) ensures that this subsystem remains unstable. We will show later on that, in general, this is true (see Lemma 4). Simple examples of matrices A_2 with complex eigenvalues that have nonzero imaginary parts are

$$A_2 = \begin{bmatrix} 0 & -a \\ a & b \end{bmatrix}, \quad \text{with} \quad |a| > |b|/2 \tag{13}$$

$$A_2 = \begin{bmatrix} -b & -a \\ c & b \end{bmatrix}, \quad \text{with} \quad ac > b^2 \tag{14}$$

and, more generally,

$$A_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with} \quad [\mathrm{Trace}(A_2)]^2 < 4\mathrm{Det}(A_2) \tag{15}$$

where a, b, c, and d are any real numbers that satisfy the respective inequalities. The eigenvalues of A_2 in Eq. (13) are complex, and they have nonzero imaginary parts; those of A_2 in Eq. (14) are purely imaginary. While matrices A_2 shown in Eqs. (13) and (14) may be considered nongeneric, the matrix A_2 in Eq. (15) is generic. Note that, while these matrices include skew-symmetric matrices, in general, they are not skew-symmetric.

Thus, a perturbation matrix P that makes the perturbed potential system described by Eq. (8) unstable is given by

$$P = \begin{bmatrix} A_2 & 0 \\ C & B \end{bmatrix} \tag{16}$$

in which A_2 is any two-by-two matrix that has a conjugate pair of complex eigenvalues with nonzero imaginary parts, and B and C are matrices that have arbitrary elements in them.

In this paper, we formalize this intuitive reasoning in a more rigorous manner, where we expand the set of positional perturbatory matrices P whose addition renders the potential system [Eq. (3)] unstable to those that are not necessarily skew-symmetric and that do not necessarily commute with K. The results provide a deeper understanding of the origin and nature of the instability caused by perturbations to potential systems that goes beyond what can be gleaned only through intuition.

II. Main Results

We begin by considering the potential system described by Eq. (3) in which the n-by-n matrix K has an eigenvalue λ whose multiplicity is m, where $2 \le m \le n$. We shall see in the following that the other eigenvalues of K will not concern us; they could be all distinct, or some of them could have multiplicities greater than unity.

Since K is symmetric, there exists an n-by-n real orthogonal matrix T such that $T^T K T = \hat{\Lambda}$, where $\hat{\Lambda}$ is a real diagonal matrix. With no loss of generality, we can write

$$\hat{\Lambda} = \begin{bmatrix} \lambda I_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix} \tag{17}$$

where we have placed all the eigenvalues of K that are distinct from λ, whatever their multiplicities, in the $(n-m)$-by-$(n-m)$ lower diagonal matrix Λ_{n-m}. Using the transformation $x(t) = Ty(t)$ in Eq. (4) and premultiplying by T^T, we get

$$\ddot{y} + \hat{\Lambda} y + \hat{P} y = 0 \tag{18}$$

where $\hat{P} = T^T P T$. The real orthogonal matrix T comprises the successive eigenvectors corresponding to the successive eigenvalues of K that lie down the diagonal of $\hat{\Lambda}$. The first m columns of T are then the (orthonormal) eigenvectors that correspond to the eigenvalue λ, and we form the n-by-m matrix

$$T_\lambda = \begin{bmatrix} t_\lambda^1 & t_\lambda^2 & \cdots & t_\lambda^p & \cdots & t_\lambda^m \end{bmatrix} \tag{19}$$

where t_λ^i, $1 \le i \le m$, are the orthonormal eigenvectors of K corresponding to the multiple eigenvalue λ. From the remainder of the (orthonormal) eigenvectors, which correspond to the successive diagonal elements of Λ [see Eq. (17)], we form the n-by-$(n-m)$ matrix

$$T_\Lambda = \begin{bmatrix} t^1 & t^2 & \cdots\cdots & t^{n-m} \end{bmatrix} \tag{20}$$

The n-by-n orthogonal matrix T can then be written as

$$T = \begin{bmatrix} t_\lambda^1 & t_\lambda^2 & \cdots & t_\lambda^p & \cdots & t_\lambda^m & t^1 & t^2 & \cdots\cdots & t^{n-m} \end{bmatrix} \tag{21}$$

We now partition this matrix T in two submatrices as follows: The first partition T_p contains the first $2 \le p \le m$ eigenvectors corresponding to the eigenvalue λ; the second partition T_r contains the remainder (hence the subscript r) of the eigenvectors. We thus have

$$T = [T_p | T_r] = [\underbrace{t_\lambda^1 \; t_\lambda^2 \cdots t_\lambda^p}_{T_p} | \underbrace{t_\lambda^{p+1} \cdots t_\lambda^m \; t^1 \; t^2 \cdots\cdots t^{n-m}}_{T_r}], \quad 2 \le p \le m \tag{22}$$

The matrix T_p is n by p, and the matrix T_r is n by $r = (n-p)$. We similarly partition the n-vector y and write $y \coloneqq [z_p^T \quad z_r^T]^T$, where $z_p = [y_1 \quad y_2 \ \cdots \ y_p]^T$ and $z_r = [y_{p+1} \quad y_{p+2} \ \cdots \ y_n]^T$. Using the partitioned matrix T given in Eq. (22), we can now write Eq. (18) as

$$\begin{bmatrix} \ddot{z}_p \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_p & 0 \\ 0 & \Lambda_r \end{bmatrix}}_{\hat{\Lambda}} \begin{bmatrix} z_p \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} T_p^T P T_p & T_p^T P T_r \\ T_r^T P T_p & T_r^T P T_r \end{bmatrix}}_{\hat{P}} \begin{bmatrix} z_p \\ z_r \end{bmatrix} = 0, \quad 2 \le p \le m \tag{23}$$

where

$$\hat{\Lambda} = \begin{bmatrix} \lambda I_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix} \coloneqq \begin{bmatrix} \lambda I_p & 0 \\ 0 & \Lambda_r \end{bmatrix} \text{so that } \Lambda_r = \begin{bmatrix} \lambda I_{m-p} & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix}, \quad 2 \le p \le m \tag{24}$$

As indicated in our intuitive reasoning given in the Introduction, the aim is now to uncouple the equation of motion of the coordinate $z_p(t)$ from that of the remainder of the system and make the response $z_p(t)$ unstable by adding a circulatory perturbation matrix. Therefore, we set 1) $\bar{N}_p \coloneqq T_p^T P T_p \ne 0$ to be a skew-symmetric matrix and 2) $T_p^T P T_r = 0$. The latter condition ensures that the coordinate $z_p(t)$ is uncoupled from the remainder of the system; the former condition ensures a circulatory perturbation to this subsystem that commutes with the potential matrix λI_p. From Eq. (23), the evolution in time of the coordinate $z_p(t)$ is then given by the equation

$$\ddot{z}_p + \lambda_1 I_p z_p + \bar{N}_p z_p = 0 \tag{25}$$

Since the matrix $\lambda_1 I_p$ commutes with the skew-symmetric matrix $\bar{N}_p$, the subsystem described by Eq. (25) is unstable and has a flutter instability [8,9]. This makes the entire system described by Eq. (23) unstable.

Later on, we will show more general matrices than $\bar{N}_p$ in Eq. (25), beyond those that are only circulatory (skew-symmetric), for which this subsystem remains unstable (see Result 2).

Remark 1: We note that this result will be true for all partitions $T = [T_p | T_r]$, where $2 \le p \le m$. □

Lemma 1: The addition of a suitable perturbation matrix P to the n-degree-of-freedom potential system described by Eq. (3) that has an eigenvalue of multiplicity m with $2 \le m \le n$ will cause the system described by Eq. (4) to become unstable and have a flutter instability if $\bar{N}_p = T_p^T P T_p \ne 0$ is a skew-symmetric matrix and $T_p^T P T_r = 0$, $2 \le p \le m$. Here, the matrix T_p contains any $2 \le p \le m$ (orthonormal) eigenvectors of K corresponding to the multiple eigenvalue λ of the matrix K, and the matrix T_r contains the remainder, $r = n - p$, of the eigenvectors. □

Remark 2: From Eqs. (23) and (25), it is clear that, from a conceptual standpoint, the resultant instability brought about when $\bar{N}_p = T_p^T P T_p \ne 0$ and $T_p^T P T_r = 0$ results from two distinct features. The first feature is the decoupling of the system of equations involving the coordinate z_p from the remainder of the system described by the coordinate z_r; this requires $T_p^T P T_r$ to be zero. The second feature ensures that the decoupled system, which corresponds to the multiple eigenvalue of K, is subjected to a nonzero circulatory force, i.e., the skew-symmetric matrix $\bar{N}_p = T_p^T P T_p \ne 0$, where $2 \le p \le m$. At root, the instability stems from the presence of the multiple eigenvalue in the potential matrix because it is this that allows the skew-symmetric matrices $\bar{N}_p$ and $\lambda_1 I_p$ to commute. □

We next show that any matrix P with $T_p^T P T_r \ne 0$ and $T_p^T P T_r = 0$ must have a specific structure.

Lemma 2: Consider a partitioned n-by-n real orthogonal matrix $T = [T_p | T_r]$, $2 \le p \le m$. An n-by-n matrix P satisfies the following conditions:

1) $T_p^T P T_p \ne 0$ is skew-symmetric.

2) $T_p^T P T_r = 0$ if, and only if,

$$P = T \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} T^T \coloneqq T \hat{P} T^T \tag{26}$$

where the p-by-p matrix $\bar{N}_p \ne 0$ is an arbitrary (real) skew-symmetric, and the matrices B_r and C_r are arbitrary (real) matrices that are r by r and r by p, respectively, with $r = n - p$. [Arbitrary matrices mean that their elements are arbitrary (real) numbers.]

Proof:

1) Assume that the matrix P is given as in Eq. (26) with the skew-symmetric matrix $\bar{N}_p \ne 0$. Then,

$$\begin{aligned} T_p^T P T_p &= T_p^T [T_p \quad T_r] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} T_p^T \\ T_r^T \end{bmatrix} T_p \\ &= [I_p \quad 0] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} T_p^T T_p \\ T_r^T T_p \end{bmatrix} \\ &= [I_p \quad 0] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} I_p \\ 0 \end{bmatrix} = [I_p \quad 0] \begin{bmatrix} \bar{N}_p \\ C_r \end{bmatrix} = \bar{N}_p \ne 0 \end{aligned} \tag{27}$$

where we have made use of the orthonormality of the columns of the matrix T in the second and third equalities. Since $\bar{N}_p$ is skew-symmetric, so is $T_p^T P T_p$.

Also, the m-by-r matrix

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$$T_p^T P T_r = T_p^T [T_p \quad T_r] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} T_p^T \\ T_r^T \end{bmatrix} T_r$$
$$= [I_p \quad 0] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} T_p^T T_r \\ T_r^T T_r \end{bmatrix}$$
$$= [I_p \quad 0] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \begin{bmatrix} 0 \\ I_r \end{bmatrix} = [I_p \quad 0] \begin{bmatrix} 0 \\ B_r \end{bmatrix} = 0 \tag{28}$$

We have thus shown that, if P has the structure given in Eq. (26) with the skew-symmetric matrix $\bar{N}_p \neq 0$, then $T_p^T P T_p \neq 0$ and $T_p^T P T_r = 0$.

2) Assume now that $T_p^T P T_p \neq 0$ is skew-symmetric, and $T_p^T P T_r = 0$.

Since the columns of the orthogonal matrix T span R^n, we can write

$$PT_p = T_p \underbrace{[a_1 \quad a_2 \ldots \ldots a_p]}_{L_p} + T_r \underbrace{[c_1 \quad c_2 \ldots \ldots c_p]}_{C_r} = T_p L_p + T_r C_r \tag{29}$$

where $a_i, i = 1, 2, \ldots, p$, are column vectors with p components; and $c_i, i = 1, 2, \ldots, p$, are column vectors with $r = n - p$ components. In the second equality, we have denoted by L_p the p-by-p (square) matrix whose columns are $a_i, i = 1, 2, \ldots, p$, and by C_r the r-by- p matrix whose columns are $c_i, i = 1, 2, \ldots, p$.

Since $T_p^T P T_p \neq 0$ is skew-symmetric, using Eq. (29), we find that

$$T_p^T P T_p = \underbrace{T_p^T T_p}_{I_p} L_p + \underbrace{T_p^T T_r}_{0} C_r = L_p \neq 0 \tag{30}$$

Furthermore, since L_p is required to be skew-symmetric, we denote it by $\bar{N}_p := L_p$.

From Eq. (29), we then get

$$PT_p = T_p \bar{N}_p + T_r C_r, \qquad \bar{N}_p \neq 0 \tag{31}$$

In a similar manner, since the columns of T span R^n, we have

$$PT_r = T_p \underbrace{[d_1 \quad d_2 \ldots d_{n-p}]}_{D_r} + T_r \underbrace{[b_1 \quad b_2 \ldots b_{n-p}]}_{B_r} = T_p D_r + T_r B_r \tag{32}$$

As before, $d_i, i = 1, 2, \ldots, n - p = r$, are column vectors with p components; and $b_i, i = 1, 2, \ldots, n - p$, are column vectors with $r = n - p$ components. The matrix D_r is p by r, and the square matrix B_r is r by r. Since $T_p^T P T_r = 0$, using Eq. (32), we get

$$T_p^T P T_r = \underbrace{T_p^T T_p}_{I_p} D_r + \underbrace{T_p^T T_r}_{0} B_r = D_r = 0 \tag{33}$$

so that Eq. (32) simplifies to

$$PT_r = T_r B_r \tag{34}$$

Combining Eqs. (31) and (34), we then have

$$P[T_p \quad T_r] = [T_p \quad T_r] \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix}, \qquad \bar{N}_p \neq 0 \tag{35}$$

Noting that $T = [T_p \quad T_r]$ is an orthogonal matrix, Eq. (35) gives

$$P = T \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} T^T \tag{36}$$

where $r = (n - p)$, $\bar{N}_p$ is an arbitrary p-by-p nonzero skew-symmetric matrix, B_r is an arbitrary r-by-r matrix, and C_r is an arbitrary r-by-p matrix. Hence, P has the structure given in Eq. (26). □

Lemma 3: If $N := P$ is circulatory (skew-symmetric), then $T_p^T N T_p \neq 0$ and $T_p^T N T_r = 0$ if, and only if,

$$N = T \begin{bmatrix} \bar{N}_p & 0 \\ 0 & \bar{N}_r \end{bmatrix} T^T \tag{37}$$

where $\bar{N}_p \neq 0$ is an arbitrary p-by-p skew-symmetric, and $\bar{N}_r$ is an arbitrary r-by-r skew-symmetric matrix $(r = n - p)$.

Proof: Setting $C_r = 0$ and $B_r = \bar{N}_r$ in Eq. (26), the "if" part of the proof is the same as that in Lemma 2. The "only if" part of the proof is also identical as in Lemma 2 until Eq. (36). From Eq. (36), if P is skew-symmetric, then $C_r = 0$ and B_r must be skew-symmetric. This proves the proposition. □

Lemma 2 leads to the following result.

Result 1: Consider the dynamical system

$$\ddot{x} + Kx = 0 \tag{38}$$

where K is an n-by-n symmetric matrix. If the potential system, regardless of its other distinct eigenvalues and their respective multiplicities, has a single eigenvalue λ with multiplicity $2 \leq m \leq n$, then there are an uncountably infinite number of perturbatory matrices P that do not, in general, commute with K and that make the system

$$\ddot{x} + (K + P)x = 0 \tag{39}$$

unstable. This flutter instability is assured for all matrices

$$P = T \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} T^T := T\hat{P}T^T \tag{40}$$

where $\bar{N}_p$ is an arbitrary nonzero square p-dimensional $(2 \leq p \leq m)$ skew-symmetric matrix, B_r is an arbitrary square matrix of dimension $r = n - p$, and C_r is an arbitrary r-by-p matrix. Here, T is the real orthogonal matrix of eigenvectors of K; and its first p eigenvectors, for any $2 \leq p \leq m$, are any p (of the m orthonormal) eigenvectors that belong to the eigenvalue λ.

Proof: From Eq. (40), we find that

$$T^T P T = \hat{P} = \begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix} \tag{41}$$

We can rewrite Eq. (23) [or, alternatively, Eq. (18)] in the notation developed as

$$\begin{bmatrix} \ddot{z}_p \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_p & 0 \\ 0 & \Lambda_r \end{bmatrix}}_{\hat{\Lambda}} \begin{bmatrix} z_p \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{N}_p & 0 \\ C_r & B_r \end{bmatrix}}_{\hat{P}} \begin{bmatrix} z_p \\ z_r \end{bmatrix} = 0 \tag{42}$$

where the matrix $\bar{N}_p$ is an arbitrary nonzero skew-symmetric matrix. The matrices B_r and C_r have (real) arbitrary elements. Hence, the matrices $\hat{\Lambda}$ and $\hat{P}$ do not commute in general. Since the matrices $K = T\hat{\Lambda}T^T$ and $P = T\hat{P}T^T$ commute if, and only if, the matrices $\hat{\Lambda}$ and $\hat{P}$ commute, the matrices K and P do not therefore commute in general. The coordinate $z_p(t)$, which is uncoupled from the remainder of the system, is unstable (flutter) since the matrices λI_p and $\bar{N}_p$ commute; and $\bar{N}_p$ is a nonzero skew-symmetric matrix [8,9]. Hence, the system described by Eq. (42) is unstable; therefore, so is the system described by Eq. (39).

Later on, we show more generally that, in Eqs. (41) and (42), instead of the skew-symmetric matrix $\bar{N}_p$, any p-by-p matrix that has at least one complex eigenvalue whose imaginary part is nonzero can be used (see Lemma 4 and Result 2). □

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Remark 3: Result 1 has a geometrical interpretation. If we think of the matrix P as a linear operator, then $\hat{P}$ is the representation of this operator in the orthonormal basis formed by the eigenvectors of K. Thus, Result 1 says that all operators whose representation in the basis comprising the eigenvectors of K is of the form $\hat{P}$ shown in Eq. (40) (with $\bar{N}_p \neq 0$) when added to the potential system will cause it to become unstable. In the basis set given by the columns of the matrix T, we see from Eq. (34) that the space spanned by the columns of T_r [see Eq. (22)] is an invariant space of the operator P. The smallest (in the sense of inclusion) such invariant space occurs when $p = m$, where m is the multiplicity of λ. In a similar fashion, when $C_r = 0$ in Eq. (41), from Eq. (31), it is seen that the space spanned by the column vectors T_p is an invariant space of the operator P, and the largest (in the sense of inclusion) such invariant space occurs when $p = m$. □

Remark 4: Only in the nongeneric case in which the eigenvalue λ of the n-by-n matrix K has multiplicity n (so that K has only one distinct eigenvalue) do the matrices $\hat{\Lambda}(K)$ and $\hat{P}(P)$ necessarily commute because, then, $\Lambda_r = \lambda I_r$ in Eq. (42) so that $\hat{\Lambda} = \lambda I_n$, and hence $\hat{\Lambda}$ commutes with all matrices $\hat{P}$. □

Remark 5: The elements of the matrices B_r and C_r are arbitrary real numbers, and hence $\hat{P}(P)$ includes skew-symmetric matrices that do, and that do not, commute with $\hat{\Lambda}(K)$. □

Since perturbations created by circulatory matrices are important from a physical standpoint, we use Lemma 3 and state this as a separate result. These circulatory matrices are a special case of the perturbation matrix P [Eq. (40)], which we denote by N (see Lemma 3).

Corollary 1: Consider the dynamical system

$$\ddot{x} + Kx = 0 \tag{43}$$

where K is an n-by-n symmetric matrix. If the potential system, regardless of its other distinct eigenvalues and their respective multiplicities, has a single eigenvalue λ with multiplicity $2 \leq m \leq n$, then there are an uncountably infinite number of circulatory matrices N that do not, in general, commute with K and that make the system

$$\ddot{x} + (K + N)x = 0 \tag{44}$$

unstable. This flutter instability is assured for all matrices

$$N = T\begin{bmatrix} \bar{N}_p & 0 \\ 0 & \bar{N}_r \end{bmatrix}T^T := T\hat{N}T^T \tag{45}$$

where $\bar{N}_p$ is an arbitrary nonzero square p-dimensional ($2 \leq p \leq m$) skew-symmetric matrix, and $\bar{N}_r$ is an arbitrary square skew-symmetric matrix of dimension $r = n - p$. Here, T is the real orthogonal matrix of eigenvectors of K; and its first p eigenvectors, for any $2 \leq p \leq m$, are any p (of the m orthonormal) eigenvectors that belong to the eigenvalue λ.

Proof: By setting $C_r = 0$ and $B_r = \bar{N}_r$ in Eq. (42), we get the equation of motion:

$$\begin{bmatrix} \ddot{z}_p \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_p & 0 \\ 0 & \Lambda_r \end{bmatrix}}_{\hat{\Lambda}}\begin{bmatrix} z_p \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{N}_p & 0 \\ 0 & \bar{N}_r \end{bmatrix}}_{\hat{N}}\begin{bmatrix} z_p \\ z_r \end{bmatrix} = 0 \tag{46}$$

The result now follows immediately from Lemma 3 by reasoning along the same lines as the proof in Result 1. □

Remark 6: In general, the matrices Λ_r and $\bar{N}_r$ do not commute, and therefore $\hat{\Lambda}$ and $\hat{N}$ do not commute. However, since $\bar{N}_r$ is an arbitrary skew-symmetric matrix in Eq. (46), it includes those matrices that do commute with Λ_r. Hence, the structure of the circulatory matrix N in Eq. (45) includes those that commute with and those that do not commute with the matrix K. Note that the matrices K and N commute if, and only if, the matrices Λ_r and $\bar{N}_r$ commute. □

Remark 7: The corresponding upper left diagonal blocks of $\hat{\Lambda}$ and $\hat{N}$ in Eq. (46) always commute with each other, i.e., λI_p and $\bar{N}_p$ always commute; it is this commutative property of the skew-symmetric matrix $\bar{N}_p$ that causes the first equation in the coordinate z_p, which is uncoupled from the coordinate z_r, to be unstable, thereby making the entire system unstable.

We now expand the set of perturbation matrices P in Eq. (40) to those that contain more general matrices than the skew-symmetric matrix $\bar{N}_p$ and yet leave the perturbed potential system unstable.

Lemma 4: Let A_p be any real p-by-p matrix that has at least one complex eigenvalue with a nonzero imaginary part. Then, the subsystem

$$\ddot{z}_p + kI_p z_p + A_p z_p = 0, \qquad 2 \leq p \leq m \tag{47}$$

is unstable.

Proof: Let $\gamma \pm i\eta$, $\eta > 0$, be a pair of complex eigenvalues of the matrix A_p. Hence, $A_p w = (\gamma + i\eta)w$, where $w \neq 0$ is an eigenvector of A_p corresponding to the eigenvalue $\gamma + i\eta$. Using the ansatz $z_p(t) = e^{\lambda t}w$ in Eq. (47), we obtain the relation

$$(\lambda^2 I_p + kI_p + A_p)w = [\lambda^2 + (k + \gamma) + i\eta]w = 0 \tag{48}$$

from which it follows that λ is complex and has a positive real part. Hence, by the ansatz, the system described by Eq. (47) is unstable (flutter). Note that A_p includes the set of nonzero skew-symmetric matrices $\bar{N}_p$. □

Lemma 4 leads to a further generalization of Result 1.

Result 2: Consider the dynamical system

$$\ddot{x} + Kx = 0 \tag{49}$$

where K is an n-by-n symmetric matrix. If the potential system, regardless of its other distinct eigenvalues and their respective multiplicities, has a single eigenvalue λ with multiplicity $2 \leq m \leq n$, then there are an uncountably infinite number of perturbatory matrices P that do not, in general, commute with K and that make the system

$$\ddot{x} + (K + P)x = 0 \tag{50}$$

unstable. This flutter instability is assured for all matrices

$$P = T\begin{bmatrix} A_p & 0 \\ C_r & B_r \end{bmatrix}T^T := T\hat{P}T^T \tag{51}$$

where A_p is an arbitrary square p-dimensional ($2 \leq p \leq m$) matrix, which has at least one complex eigenvalue whose imaginary part is nonzero; B_r is an arbitrary square matrix of dimension $r = n - p$; and C_r is an arbitrary r-by-p matrix. Here, T is the real orthogonal matrix of eigenvectors of K; and its first p eigenvectors, for any $2 \leq p \leq m$, are any p (of the m orthonormal) eigenvectors that belong to the eigenvalue λ.

Proof: Replacing $\bar{N}_p$ by A_p in Eq. (42), we get

$$\begin{bmatrix} \ddot{z}_p \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_p & 0 \\ 0 & \Lambda_r \end{bmatrix}}_{\hat{\Lambda}}\begin{bmatrix} z_p \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} A_p & 0 \\ C_r & B_r \end{bmatrix}}_{\hat{P}}\begin{bmatrix} z_p \\ z_r \end{bmatrix} = 0 \tag{52}$$

Using Lemma 4, and noting that the matrix A_p makes the subsystem in Eq. (47) unstable, the result follows. □

Examples of A_p when $p = 2$ are given in Eqs. (13–15). When $2 < p \leq m$, a simple p-by-p matrix A_p that has at least one complex eigenvalue whose imaginary part is nonzero is

$$A_p = \begin{bmatrix} A_2 & 0 \\ F_j & E_j \end{bmatrix} \tag{53}$$

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in which $j = p - 2$, and A_2 is any two-by-two real matrix that has a pair of complex conjugate eigenvalues whose imaginary parts are nonzero. The matrix F_j is j-by-2, and E_j is j-by-j; both of these matrices have arbitrary (real) elements.

Replacing $\bar{N}_p$ by the matrix A_p [Eq. (51)] in Lemmas 1 and 2, we get the following more general results whose proofs follow the same lines as those in Lemma 1 and Lemma 2.

Lemma 5: The addition of a suitable perturbation matrix P to the n-degree-of-freedom potential system described by Eq. (3) that has an eigenvalue of multiplicity m with $2 \le m \le n$ will cause the system described by Eq. (4) to become unstable and have a flutter instability if $A_p = T_p^T P T_p$ is a matrix that has at least one eigenvalue with a nonzero imaginary part, and $T_p^T P T_r = 0$, $2 \le p \le m$. Here, the matrix T_p contains any $2 \le p \le m$ (orthonormal) eigenvectors of K corresponding to the multiple eigenvalue λ of the matrix K, and the matrix T_r contains the remainder, $r = n - p$, of the eigenvectors. □

Lemma 6: Consider a partitioned n-by-n real orthogonal matrix $T = [T_p | T_r]$, $2 \le p \le m$. An n-by-n matrix P satisfies the following conditions:

1) $T_p^T P T_p$ is any p-by-p matrix that has at least one complex eigenvalue with a nonzero imaginary part.

2) $T_p^T P T_r = 0$ if, and only if,

$$P = T \begin{bmatrix} A_p & 0 \\ C_r & B_r \end{bmatrix} T^T := T \hat{P} T^T \tag{54}$$

where the p-by-p matrix A_p is an arbitrary (real) matrix with at least one complex eigenvalue whose imaginary part is nonzero, and the matrices B_r and C_r are arbitrary (real) matrices that are r by r and r by p, respectively, with $r = n - p$. □

It is useful to particularize Result 2 for the case when $p = 2$, no matter what the multiplicity of the repeated eigenvalue λ of the potential matrix K (see Remark 9), and no matter what its other eigenvalues and their respective multiplicities are (Remark 9).The following section uses this result.

Corollary 2: If the potential system

$$\ddot{x} + Kx = 0 \tag{55}$$

where K is an n-by-n matrix, has an eigenvalue with multiplicity two (or more), then this system can be made unstable by the addition of an uncountably large number of infinitesimal positional perturbatory matrices P that do not commute with K.

Proof: Assume that the matrix K has an eigenvalue λ with multiplicity $2 \le m < n$. Let T be a real orthogonal matrix such that $K = T\Lambda T^T$. With no loss of generality, we can write

$$\Lambda = \mathrm{diag}(\lambda, \lambda, \lambda_3, \lambda_4, \ldots, \lambda_n) \tag{56}$$

A total of $(m-2)$ eigenvalues in the list $\lambda_3, \lambda_4, \ldots, \lambda_n$ in Eq. (56) has the value λ. Using the transformation $x(t) = Ty(t)$, we obtain the equation

$$\begin{bmatrix} \ddot{z}_2 \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_2 & 0 \\ 0 & \Lambda_{n-2} \end{bmatrix}}_{\hat{\Lambda}} \begin{bmatrix} z_1 \\ z_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{57}$$

where the 2-vector $z_2(t) = [y_1, y_2]^T$, $\Lambda_{n-2} = \mathrm{diag}(\lambda_3, \lambda_4, \ldots, \lambda_n)$, and $z_r(t) = [y_3, y_4, \ldots, y_n]^T$. The addition of a perturbatory matrix of the form

$$\hat{P} = \begin{bmatrix} A_2 & 0 \\ C_{n-2} & B_{n-2} \end{bmatrix} \tag{58}$$

where A_2 is an arbitrary (nonzero) two-by-two matrix that has a complex eigenvalue whose imaginary part is nonzero, B_{n-2} is any arbitrary square matrix of dimension $n-2$, and C_{n-2} is an arbitrary matrix of dimension $(n-2)$-by-2, will cause the system

$$\begin{bmatrix} \ddot{z}_2 \\ \ddot{z}_r \end{bmatrix} + \underbrace{\begin{bmatrix} \lambda I_2 & 0 \\ 0 & \Lambda_{n-2} \end{bmatrix}}_{\hat{\Lambda}} \begin{bmatrix} z_2 \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} A_2 & 0 \\ C_{n-2} & B_{n-2} \end{bmatrix}}_{\hat{P}} \begin{bmatrix} z_2 \\ z_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{59}$$

to be unstable since the two-degree-of-freedom uncoupled subsystem described by the coordinate $z_2(t)$ is unstable, as shown in Lemma 4.

For simplicity, one could use a two-by-two skew-symmetric matrix $\bar{N}_2 \neq 0$ for A_2 in Eq. (59) or those given in Eqs. (13–15). The matrices $\hat{\Lambda}(K)$ and $\hat{P}(P)$ do not commute, in general, since B_{n-2} and C_{n-2} have arbitrary elements.

Were we to be interested only in circulatory perturbations, we would set $C_{n-2} = 0$ and choose the elements of A_2 and B_{n-2} so that they were skew-symmetric.

Comparing Eq. (16) with the matrix $\hat{P}$ given in Eq. (58), we see that what we had intuited in the Introduction section is indeed true. □

Remark 8: Clearly, all that is needed for the positional perturbatory matrix P to make the perturbed system given in Eq. (59) unstable in flutter is for the potential matrix K to have just one multiple eigenvalue with only a multiplicity of two. □

Remark 9: On restricting the perturbatory matrix in Eq. (51) to the case when $C_r = 0$, the operator $\hat{P}$ can be thought of as the direct sum of the operators A_p and B_r so that we have

$$\hat{P} = A_p \oplus B_r \tag{60}$$

where A_p is any p-by-p matrix that has at least one pair of complex conjugate eigenvalues with nonzero imaginary parts, and B_r is an arbitrary r-by-r matrix $(r = p - n)$. The matrix $P = T\hat{P}T^T$ would then make the system $\ddot{x} + (K + P)x = 0$ unstable.

Restricting A_p to being skew-symmetric so that $\bar{N}_p := A_p$, Eq. (60) reduces to

$$\hat{P} = \bar{N}_p \oplus B_r \tag{61}$$

A further restriction on B_r to a skew-symmetric matrix $\bar{N}_r := B_r$ gives

$$\hat{N} := \hat{P} = \bar{N}_p \oplus \bar{N}_r \tag{62}$$

and we now get the result obtained for circulatory perturbations.[‡]. □

Remark 10: We can use the arguments developed earlier in this paper for other distinct eigenvalues besides λ whose multiplicities are greater than one if the potential system has such eigenvalues (see the Example section). □

Remark 11: Based on Result 2, we have the following observation. For any n-by-n orthogonal matrix $T = [T_p | T_{n-p}]$, $2 \le p \le m$, and any two real n-by-n matrices $(m \le n)$,

$$\hat{\Lambda} = \mathrm{diag}(\lambda I_p, \Lambda), \quad \text{and} \quad \hat{P} = \begin{bmatrix} A_p & 0 \\ C_r & B_r \end{bmatrix}, \quad 2 \le p \le m \tag{63}$$

where Λ is a diagonal matrix, A_p is a p-by-p arbitrary matrix that has at least one complex eigenvalue with a nonzero imaginary part B_r, $r = n - p$ is an r-by-r matrix with arbitrary elements, and C_r is an r-by-p matrix with arbitrary elements, the perturbed potential system

$$\ddot{x} + (K + P)x = 0 \tag{64}$$

with $K = T\hat{\Lambda}T^T$ and $P = T\hat{P}T^T$ is unstable and has a flutter instability. The matrices K and P do not commute in general. □

[‡]Personal communication with J. Awrejcewicz of the Lodz University of Technology in Lodz, Poland on 13 March 2018.

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III. Example

Consider the potential system described by the equation

$$\ddot{x} + \hat{\Lambda} x = 0 \tag{65}$$

where the potential matrix

$$\hat{\Lambda} = \mathrm{diag}(a, a, a, b, b, c, d) \tag{66}$$

with $a \neq b \neq c \neq d$.

We start by considering the addition of circulatory forces (matrix) to this system so that the perturbed system is described by

$$\ddot{x} + \hat{\Lambda} x + \hat{N} x = 0 \tag{67}$$

The structure of the various (infinitesimal) circulatory matrices $\hat{N}$ that, in general, do not commute with $\hat{\Lambda}$ and that will cause this perturbed potential system to become unstable are shown in Fig. 1. The basic idea is to 1) uncouple a potential subsystem, or part thereof, with multiple eigenvalues from the remainder of the potential system; then 2) add a nonzero skew-symmetric matrix (perturbation) to this uncoupled subsystem while still keeping this subsystem uncoupled from the remainder of the system; and finally 3) add an arbitrary circulatory perturbation to the remainder of the system, keeping it uncoupled from the subsystem chosen in Eq. (1). As stated before, when the matrices $\hat{N}$ and $\hat{\Lambda}$ do not commute, the matrices K and N also do not commute.

We show in Fig. 1 the potential system defined by the matrix $\hat{\Lambda}$ and the structure of the circulatory matrices $\hat{N} = \hat{N}_i$, $i = 1, 2 \ldots, 6$, that when used in Eq. (67) ensure instability in this perturbed potential system; the elements of matrices with these structures can be chosen so that they do not commute with $\hat{\Lambda}$ and yet make the system described by Eq. (67) unstable. For now, arbitrary nonzero skew-symmetric matrices are shown by slanted hatched lines, and arbitrary skew-symmetric matrices are shown by vertical hatched lines. By "arbitrary," we mean that the matrix elements are arbitrary real numbers.

The structure of the matrix $\hat{N}_1$ is such that it isolates the upper two-degree-of-freedom subsystem from the potential system (matrix $\hat{\Lambda}$). Although the multiplicity of the eigenvalue a is three in the matrix $\hat{\Lambda}$, this two-degree-of-freedom system is thought of as a subsystem with a multiple eigenvalue a whose multiplicity is two (see Fig. 1b). To ensure the (flutter) instability of this two-degree-of-freedom potential subsystem, and therefore instability of the entire seven-degree-of-freedom circulatory system shown in Eq. (67), when perturbed by a circulatory matrix, an arbitrary nonzero two-by-two skew-symmetric matrix (shown by slanting hatched lines) is used while ensuring that this subsystem remains uncoupled from the rest of the system. The diagonal block shown with vertical hatching is any square skew-symmetric matrix whose elements are arbitrary real numbers. Hence, in general, the matrices $\hat{\Lambda}$ and $\hat{N}_1$ do not commute. However, the set of matrices that have the structure shown in $\hat{N}_1$ also includes all the skew-symmetric matrices that do commute with $\hat{\Lambda}$. For example, were this arbitrary five-by-five skew-symmetric lower diagonal block (with vertical hatching) chosen to be zero, then, indeed the two matrices $\hat{\Lambda}$ and $\hat{N}_1$ would commute.

Similarly, the structure of the matrix $\hat{N}_2$ considers the upper three-degree-of-freedom subsystem from the potential system (matrix $\hat{\Lambda}$) with the multiple eigenvalue a with multiplicity three (see Fig. 1c). To ensure instability of this three-degree-of-freedom potential circulatory subsystem, it has an arbitrary nonzero three-by-three skew-symmetric matrix (shown by slanting hatched lines) while ensuring that this subsystem remains uncoupled from the rest of the system. The diagonal block shown in Fig. 1c with vertical hatching is any arbitrary skew-symmetric matrix whose addition to the potential system ($\hat{\Lambda}$ matrix) continues to leave the three-by-three subsystem uncoupled from the rest of the dynamics. Again, $\hat{\Lambda}$ and $\hat{N}_2$ do not commute, in general. But, as before, if we choose the vertically hatched matrix in the lower diagonal block of $\hat{N}_2$ to be the zero matrix, then they would. The set of matrices $\hat{N}_2$ that can be used includes all those that commute with $\hat{\Lambda}$.

When the lower two-degree-of-freedom subsystem from the potential system (matrix $\hat{\Lambda}$) with the multiple eigenvalue a with multiplicity two is chosen as the subsystem (see Fig. 1d), to ensure its instability, the structure of the matrix $\hat{N}_3$ has the two-by-two nonzero skew-symmetric matrix block (slanted hatching) so that this block is uncoupled from the rest of the dynamics of the system. The matrix $\hat{\Lambda}$ is shown again for convenience in Fig. 1a at a level with the matrix $\hat{N}_3$ so that this two-degree-of-freedom system, which is different from the one used in $\hat{N}_1$, can be easily identified. The remainder of the diagonal blocks (shown by vertical hatched lines) of the matrix $\hat{N}_3$ are skew-symmetric. The one-by-one block in the upper left corner is therefore just a zero, as shown.

As pointed out in Remark 10, the structure of $\hat{N}_4$ shown in Fig. 1e permits the two-degree-of-freedom subsystem from the potential system (matrix $\hat{\Lambda}$) with the multiple eigenvalue b that has multiplicity two (see Fig. 1e) to be made unstable. To ensure this potential circulatory subsystem's instability, $\hat{N}_4$ has the corresponding arbitrary two-by-two

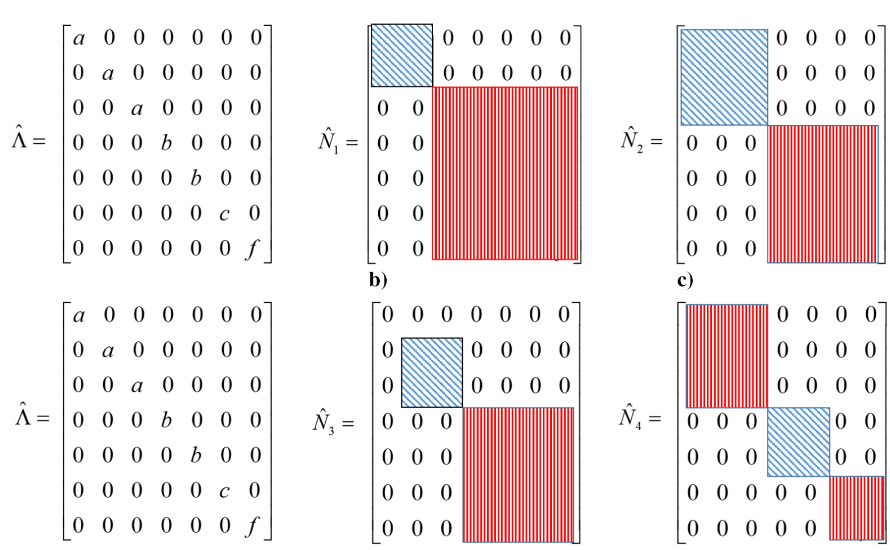

Fig. 1 Structure of circulatory perturbations of potential system (65) that will make the perturbed potential system unstable. Slanted hatching shows nonzero skew-symmetric matrices, and vertical hatching shows arbitrary skew-symmetric matrices.

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nonzero skew-symmetric matrix (shown by slanted hatching in Fig. 1e), and this block is uncoupled from the remainder of the dynamics of the system. The other skew-symmetric diagonal blocks shown by vertical hatchings contain arbitrary elements. The set of matrices with the structure shown in $\hat{N}_4$ also includes matrices that commute with $\hat{\Lambda}$. For example, were the vertically hatched blocks chosen to be zero matrices, then the matrices $\hat{\Lambda}$ and $\hat{N}_4$ would commute. They would also commute if the upper vertically hatched block is any skew-symmetric matrix and the lower vertically hatched block is the zero matrix. The skew-symmetric matrix $\hat{N}_4$ includes all block diagonal skew-symmetric matrices that commute with $\hat{\Lambda}$.

A more general structure $\hat{N}_5$ than that shown by $\hat{N}_4$ for the skew-symmetric matrix that keeps this chosen two-degree-of-freedom circulatory subsystem unstable (and decoupled from the rest of the system) is shown in Fig. 2a. Here, the additional matrices S and $-S^T$ shown by the vertical hatched lines have arbitrary elements. When $S \neq 0$, $\hat{N}_5$ does not commute with $\hat{\Lambda}$.

In a similar way, instead of using the matrix structure $\hat{N}_3$ shown in Fig. 1d, we could have used (as shown in Fig. 2b) a more general skew-symmetric matrix structure (denoted by $\hat{N}_6$) that makes the same two-degree-of-freedom subsystem as in Fig. 1d with the multiple eigenvalue a (that has multiplicity two) unstable, just as the matrix $\hat{N}_3$ has. Here, the column vector e shown by the vertical hatched lines is arbitrary. When $e \neq 0$, $\hat{N}_6$ does not commute with $\hat{\Lambda}$.

As stated in Remark 10, for any orthogonal matrix T, consider the potential system $\ddot{x} + Kx = 0$ with $K = T\hat{\Lambda}T^T$, where $\hat{\Lambda}$ has the structure given in Eq. (66). Addition of a circulatory (infinitesimal) perturbation to this system given by the skew-symmetric matrices $N = T\hat{N}_iT^T$ where the matrices $\hat{N}_i$, $i = 1, 2, \ldots, 6$, have any of the structures discussed earlier in this paper and shown in Figs. 1 and 2 will cause the potential circulatory system $\ddot{x} + (K + N)x = 0$ in which the matrices K and N do not commute (in general) to be unstable.

Equation (61) states that we can also have perturbatory matrices P that need not be circulatory to create instability. Instead of using nonzero skew-symmetric matrices that have arbitrary real elements in them, shown by the vertical hatching in Figs. 1 and 2, we could use any arbitrary matrix (of the proper size) wherever the vertical hatchings are shown in these figures. The perturbed potential system $\ddot{x} + (K + P)x = 0$ will remain unstable because of the uncoupled subsystems that belong to the potential matrix that are made unstable by the presence of the nonzero skew-symmetric matrices that are shown by the slanted hatchings in Figs. 1 and 2. For example, instead of using the circulatory matrix structure with $\hat{N} = \hat{N}_5$ (see Fig. 2a) in Eq. (67) to make the perturbed potential system unstable, we could use the matrix structure $\hat{P}_5$ shown in Fig. 3a, which need not be circulatory, and which does not commute with $\hat{\Lambda}$ in general; yet, it guarantees instability of the perturbed potential system. The vertically hatched area in Fig. 3a now contains any arbitrary matrix elements; the slanted hatched area contains the two-by-two nonzero skew-symmetric matrix, as in $\hat{N}_5$ in Fig. 2a. Or, as in Eq. (41), one could use an even more general perturbatory matrix structure $\hat{P}$, shown in Fig. 3b, which again need not commute with $\hat{\Lambda}$ and which makes the perturbed potential system unstable. Elements of the matrix in the vertically hatched area are arbitrary, and the slanted hatched area has a skew-symmetric matrix.

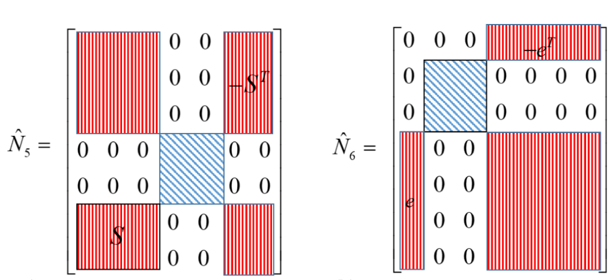

Fig. 2 Alternative structures for a) $\hat{N}_4$ and b) $\hat{N}_3$.

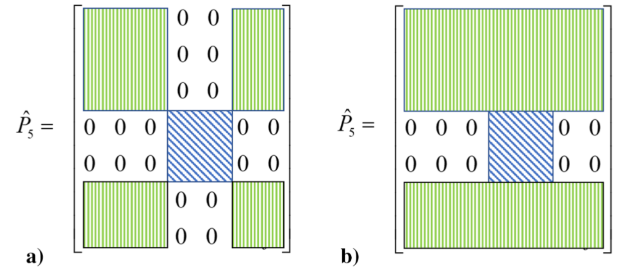

Fig. 3 Vertical hatched area is filled by any arbitrary matrix elements. Slanted hatched area is filled by *any* nonzero skew-symmetric matrix or any matrix that has an eigenvalue with a nonzero imaginary part.

Finally, while we have used nonzero skew-symmetric matrices shown by the slanted hatched regions in Figs. 1–3, by Result 2, all these skew-symmetric matrices $(\bar{N}_p)$ in these regions in all three figures can each be replaced by matrices A_p of the appropriate dimensions, which have at least one pair of complex conjugate eigenvalues with a nonzero imaginary part [see Eqs. (13–15), and (53)]. See Fig. 3.

This example shows some of the explicit structures that the perturbatory matrices can have and the routes to guarantee that the potential system [Eq. (65)] when perturbed by them is assuredly unstable. It shows that there are an uncountably infinite number of positional perturbatory matrices $P = T\hat{P}T^T$, both circulatory and noncirculatory, which do and do not commute with $K = T\hat{\Lambda}T^T$, that can perturb a potential system and make it unstable if the potential matrix has a single repeated eigenvalue with multiplicity of two (or more).

More generally, routes to instability in like manner can be tailored (through the use of appropriate matrices $\hat{P}$ as pointed out in Result 1 and in Corollary 1) for any n-by-n potential matrix that has just a single multiple eigenvalue with a multiplicity of only two. There are thus both circulatory and noncirculatory matrices that can be used as positional perturbations to such a potential matrix to ensure its instability when so perturbed. There is no requirement that these perturbatory matrices must commute with the potential matrix.

IV. Engineering Application of the Mathematical Results

The aforementioned results show that, if a stable potential system has even a single eigenvalue that has just a multiplicity of two, the perturbed potential system can be made unstable by 1) isolating the two-degree-of-freedom potential subsystem that has the multiple eigenvalue from the remainder of the system; 2) adding an arbitrary nonzero perturbing circulatory force (matrix) to this subsystem, keeping it isolated from the remainder of the system; and 3) adding positional forces to the remainder of the potential system pretty much as we please but ensuring that these added forces keep the dynamics of the perturbed two-degree-of-freedom subsystem (in steps 1 and 2) uncoupled from that of the rest of the system (see Fig. 3b, for example). [For simplicity, in the aforementioned step 2, we have used a circulatory force (matrix). Instead of a circulatory matrix, any two-by-two matrix with a complex eigenvalue whose imaginary part is nonzero would do as well; see Eq. (58).]

We see that the critical element in creating instability in a stable potential system is the presence of a multiple eigenvalue of the potential matrix. If such a multiple eigenvalue exists, then even an infinitesimal "whiff" of a circulatory matrix appropriately added to the potential system, as in step 2, makes it unstable. Thus, in order to evaluate the practical and engineering impact of these mathematizations, we need to ask the following question: Do multiple eigenvalues actually occur in naturally occurring and in engineered systems?

Multiple eigenvalues (frequencies of vibration) do occur in natural and engineered systems, especially when a physical system shows considerable symmetry or when it is constrained in special ways. From a practical viewpoint, it occurs in spacecraft systems, tall building vibrations caused by strong earthquake ground shaking, and automotive systems. For example, the fourth bending mode could have the same frequency as the second torsional mode of

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vibration in a tall building structure or a spacecraft structure. Multiple frequencies of vibration also arise commonly in automotive structures. Often, when complex systems are modeled and simulated by hundreds, and often thousands, of degrees of freedom, there is a likelihood of obtaining one or more frequencies of vibration that have multiplicities of two or more. Such computational results at times do show "identical" frequencies, although they may not be mathematically (exactly) so because of limitations of computational accuracy.

However, the foregoing mathematizations ask for two frequencies of vibration to be mathematically (exactly) equal; to see if this requirement is really too stringent when dealing with real-life dynamical systems, it behooves us to investigate what might happen when two frequencies of vibration are close, or very close, but not mathematically (exactly) the same. For, after all, that two frequencies of a physical system are the same is generally assessed by experiments; and experiments have limits to the accuracy of their findings. In other words, from a practical perspective, one would need to know if the recipe provided in the aforementioned three steps still engenders instability in a potential system even when two frequencies of vibration are "close" to one another or are deemed to be the "same" within known experimental error bounds. And, then, the obvious question arises: How does one define close?

At the core, the instability is generated by considering just the two-degree-of-freedom subsystem (a component of the entire potential system) that has a multiple eigenvalue. It is then enough to explore subsystems like those described in Eq. (9) in which the potential system has a multiple eigenvalue of multiplicity two and for which we know that an infinitesimal value of α (a whiff of a circulatory perturbing force) will make the subsystem unstable. See Fig. 1a. Our questions are then the following: Would this continue to be the case even when the eigenvalues are not exactly the same but close? How close?

Hence, from practical considerations, we are led to consider the stability of the two-degree-of-freedom potential subsystem

$$\underbrace{\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}}_{\ddot{z}_2}+\underbrace{k\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{kI_2}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{z_2}+\begin{bmatrix}\varepsilon&0\\0&0\end{bmatrix}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{z_2}+\underbrace{\begin{bmatrix}0&\alpha\\-\alpha&0\end{bmatrix}}_{A_2}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{z_2}=0,\qquad \alpha\neq 0 \tag{68}$$

in which two eigenvalues of the potential matrix are $k+\varepsilon$, $k>0$.

The characteristic polynomial $p(\lambda)$ of this system is $\lambda^4+(2k+\varepsilon)\lambda^2+\alpha^2+\varepsilon k+k^2$. Setting $\mu=\lambda^2$ in the biquadratic, we obtain the quadratic equation

$$\mu^2+(2k+\varepsilon)\mu^2+\alpha^2+\varepsilon k+k^2=0 \tag{69}$$

whose roots are

$$\mu_{1,2}=\lambda_{1,2}^2=-\frac{(2k+\varepsilon)}{2}\pm\frac{1}{2}\sqrt{\varepsilon^2-4\alpha^2}$$

When the roots $\mu_{1,2}$ are complex, the characteristic polynomial $p(\lambda)$ of Eq. (68) has a root with a positive real part and the subsystem described by Eq. (68) is unstable (flutter). Clearly, when $\varepsilon=0$ and the eigenvalues of the potential matrix are both (exactly) equal to k, only an infinitesimal value of α would suffice to cause instability since $\mu_{1,2}$ would then be complex numbers. When $\varepsilon\neq 0$, $\mu_{1,2}$ is complex as long as

$$4\alpha^2>\varepsilon^2,\quad\text{or}\quad |\alpha|>\frac{|\varepsilon|}{2} \tag{70}$$

and the instability of Eq. (68) is thereby assured.

We nondimensionalize Eq. (68) by using the transformations $x_1(t)=\bar{x}_1x_1^*(t)$, $x_2(t)=\bar{x}_2x_2^*(t)$, and $\tau=\sqrt{k}t$, where 1) $x_1^*(t)$, $x_2^*(t)$, and τ are dimensionless; and 2) $\bar{x}_1$ and $\bar{x}_2$ are constants with dimensions. We thus obtain the dimensionless equations

$$\begin{bmatrix}x_1^{*\prime\prime}\\x_2^{*\prime\prime}\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}+\frac{1}{k}\begin{bmatrix}\varepsilon&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}+\frac{1}{k}\begin{bmatrix}0&\alpha\\-\alpha&0\end{bmatrix}\begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}=0 \tag{71}$$

where the primes now indicate differentiation with respect to the dimensionless time τ. Let ω_1 and ω_2 be the two dimensionless frequencies of vibration of the two-degree-of-freedom (unperturbed) potential system. Then, $\omega_1^2=1+\varepsilon/k$, and $\omega_2^2=1$. Thus, the discrepancy in the dimensionless frequency is given by

$$\rho:=\omega_1-\omega_2=\sqrt{1+\frac{\varepsilon}{k}}-1 \tag{72}$$

so that

$$|\varepsilon|=2k\left|\rho+\frac{1}{2}\rho^2\right| \tag{73}$$

From Eq. (71), a measure of the (dimensionless) magnitude of the circulatory force can be taken to be $|\theta|:=|\alpha|/k$. Equation (70) then informs us that, when

$$|\theta|>\left|\rho+\frac{1}{2}\rho^2\right| \tag{74}$$

the system in Eq. (71) is guaranteed to be unstable. We note that the "remainder" of the system, of which the subsystem in Eq. (68) is a part (and which we have not considered here), might also be unstable and with even smaller perturbational forces; Eq. (74) thus gives only a sufficient condition to make the entire system unstable.

One way of restating this is the following: When the dimensionless frequencies of the (uncoupled) two-degree-of-freedom potential subsystem, which is used to bring about destabilization, are not exactly the same and differ by $|\rho|\ll 1$, subsystem (71) is made assuredly unstable by a dimensionless circulatory perturbation measured (in magnitude) by $|\theta|$, whose magnitude is given in Eq. (74). The magnitude of the circulatory perturbation $|\theta|$ required to cause instability therefore increases with the (dimensionless) discrepancy $|\rho|$ from equality between the two frequencies. Hence, a small, or minute, difference $O(|\rho|)$ between the frequencies requires a small, or minute, circulatory perturbation $O(|\theta|)$ that guarantees instability of the subsystem, and therefore of the entire potential system. As pointed out in Result 2, the perturbatory matrix for the entire system need not be circulatory, and it can have the general form shown in Eq. (63) (see Fig. 3b).

From a practical standpoint, Eq. (74) provides a result that has relevance to real-life systems that have two close, although not identical, frequencies of vibration. It shows that such systems can be dangerously susceptible to instabilities created by small positional perturbations. The closer the two frequencies get to being identical, the smaller the circulatory perturbations needed to cause these flutter instabilities.

V. Conclusions

The paper's main contribution is the central result that stable potential systems whose matrices have one or more eigenvalues with multiplicity greater than one can lose their stability under infinitesimal positional perturbations that need not be circulatory and that need not commute with the potential matrix. Routes to instability that rely on keeping a subsystem of such a potential system unstable and uncoupled from its remainder are demonstrated. Both finite and infinitesimal perturbatory matrices are included. This leads to the development of the explicit structure of general matrices, and not just circulatory matrices, that are not required to commute with potential matrices and that guarantee instability when such potential systems are perturbed by them. These general perturbatory matrices that guarantee instability encompass the set of circulatory matrices that commute with, and that do not commute with, the potential matrix.

Results obtained to date state that, when the circulatory matrix commutes with the potential matrix, the potential system when perturbed by such a circulatory matrix can be made unstable using arbitrarily small circulatory matrices [8,9]. The property that two matrices commute is a fairly strong requirement, and the paper shows that this requirement can be eliminated. It uses the underlying fundamental reason for this (mathematical) commutation result that the potential matrix has one or more multiple eigenvalues [9]. It is shown that, if the potential matrix has even a single repeated eigenvalue with just a multiplicity of two, there exists an uncountably infinite set of general positional perturbatory matrices that make the perturbed potential system unstable. Circulatory matrices that do and do not commute with the potential matrix and cause the perturbed potential system to be unstable are special cases of these general perturbatory matrices.

The paper thus significantly expands previous generalizations of Merkin's result [1] by including in the set of circulatory perturbations (matrices) that cause instability in potential systems those that do not commute with the potential matrix and are not circulatory. It goes beyond presently known results to more general (noncirculatory) perturbatory matrices while simultaneously eliminating any commutation requirements.

Practical engineering takeaways from the mathematical results developed are provided. When two frequencies of vibration of a potential system are (exactly) identical, an uncountably infinite set of infinitesimal perturbations of the potential system cause instability. When experimental and/or simulation results on real-life systems show a small, or minute, discrepancy between two frequencies of vibration, then a uncountably infinite set of correspondingly small, or minute, perturbations guarantee instability of the entire potential system. And, what is exactly meant by small discrepancies and small perturbations is precisely defined.

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